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On intuitionistic fuzzy prime and $\mathcal{G}$-prime ideals of a near ring

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Abstract. Group actions are a valuable tool for investigating the symmetry and automorphism features of near-rings. In this paper, we introduced group action on intuitionistic fuzzy (IF) ideals of near ring $\mathcal{N}$. We defined intrinsic product of IF ideals of $\mathcal{N}$ and investigated some properties of IF prime ideals under group action on it. Moreover, we developed an idea of IF $\mathcal{G}$-prime ideal of $\mathcal{N}$. Moreover, we have shown that for an IF ideal $\mathcal{F}$ of $\mathcal{N}$ such that $\mathcal{F}^{\mathcal{G}} = \bigcap_{g\in\mathcal{G}} \mathcal{F}^g$, then $\mathcal{F}^{\mathcal{G}}$ is the largest $\mathcal{G}$-invariant IF ideal of $\mathcal{N}$ contained in $\mathcal{F}$. We additionally prove that $\mathcal{G}$-primeness of $\mathcal{F}^{\mathcal{G}}$ is uniquely determined by $\mathcal{G}$-primeness of $\mathcal{F}$ upto $\mathcal{G}$-orbits.

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1. Introduction

In order to counteract ambiguity in daily life, Zadeh [1] extended the idea of classical set theory by introducing the fuzzy set. Many direct and indirect generalizations of the fuzzy set have been developed and effectively used in the majority of real-world problems. Pattern recognition, decision-making issues, clustering analysis, and medical diagnostics are just a few of the real-world applications where the FS theory has been researched. Inadequate knowledge of the function's negative membership degree has, regrettably, led to the failure of the FS theory. In order to solve this issue, Atanassov [2] included the negative membership degree of the function in FS theory in such a way that sum of the positive membership degree and negative membership degree must not exceed one. Liu [3] has studied fuzzy ideals of a ring and many researchers extended this concept. Kim and Kim [4] gave the notion of fuzzy ideals of near rings. The idea of "Intuitionistic Fuzzy set" as a generalization of fuzzy sets was given by Atanassov [2] in 1986. Biswas [5] studied IF subgroups of a group using the concept of IF sets and extended to group theory. The concepts of IF prime ideals and IF weak prime ideals in ring were developed by Hur et al. [6] in a ring. Jun and Park [7] introduced the concept of IF $\mathcal{N}$-subgroups of a near ring. The features of IF ideals of near rings were also addressed by Jianming and Xueling [8].

Very recently Asma Ali et al. introduced and studied group action on fuzzy ideals of a near ring [9]. Lee and Park [10] studied group action on the IF ideals of a ring $\mathcal{R}$ and derived a relationship between the IF $\mathcal{G}$-prime ideals of $\mathcal{R}$ and the IF prime ideals of $\mathcal{R}$. In this paper we define group action on an IF ideals of near ring $\mathcal{N}$ and study $\mathcal{G}$-invariant IF ideals of $\mathcal{N}$, intrinsic products of IF ideals and $\mathcal{G}$-primeness of IF ideals of $\mathcal{N}$. Hence extend the results of [10] in case of near rings.

2. Preliminaries

For basic definitions of fuzzy ideals and anti fuzzy ideals of a near ring one may be referred to [11].

Definition 2.1. Let $\mathcal{Z}$ be a nonempty set. Then an *intuitionistic fuzzy set* (briefly, IF se)t $\mathcal{M}$ in Z has a form $\mathcal{M} = \{(\mathfrak{z}, \eta_{\mathcal{M}}(\mathfrak{z}), \delta_{\mathcal{M}}(\mathfrak{z}))\}$, where the functions $\eta_{\mathcal{M}}, \delta_{\mathcal{M}} : \mathcal{Z} \to [0,1]$ signify the degree of membership and non membership respectively and $0 \leq \eta_{\mathcal{M}}(\mathfrak{z}) + \delta_{\mathcal{M}}(\mathfrak{z}) \leq 1, \mathfrak{z} \in \mathcal{Z}$.

For our simplicity we use $\mathcal{M} = (\eta_{\mathcal{M}}, \delta_{\mathcal{M}})$ for IF set $\mathcal{M} = \{(\mathfrak{z}, \eta_{\mathcal{M}}(\mathfrak{z}), \delta_{\mathcal{M}}(\mathfrak{z}))|\mathfrak{z} \in \mathcal{Z}\}$ in $\mathcal{Z}$.

Definition 2.2. An IF set $\mathcal{A} = (\eta_{\mathcal{A}}, \delta_{\mathcal{A}})$ in a near ring $\mathcal{N}$ is called an *intuitionistic fuzzy ideal* (briefly, IF ideal) of $\mathcal{N}$, if $\eta_{\mathcal{A}}$ and $\delta_{\mathcal{A}}$ are fuzzy ideal and anti fuzzy ideal respectively.

Lemma 2.3. *Let $\mathcal{M} = (\eta_{\mathcal{M}}, \delta_{\mathcal{M}})$ and $\mathcal{N} = (\eta_{\mathcal{N}}, \delta_{\mathcal{N}})$ be IF sets in a set $\mathcal{S}$. Then we define:*

(1) $\mathcal{M} \subseteq \mathcal{N} \iff (\forall s \in \mathcal{S})(\eta_{\mathcal{M}}(s) \leq \eta_{\mathcal{N}}(s), \quad \delta_{\mathcal{M}}(s) \geq \delta_{\mathcal{N}}(s))$,
(2) $\mathcal{M} = \mathcal{N} \iff \mathcal{M} \subseteq \mathcal{N}$ *and* $\mathcal{N} \subseteq \mathcal{M}$,
(3) $\mathcal{M} \cap \mathcal{N} = (\eta_{\mathcal{M}} \bigwedge \eta_{\mathcal{N}}, \quad \delta_{\mathcal{M}} \bigvee \delta_{\mathcal{N}})$,
(4) $\mathcal{M} \cup \mathcal{N} = (\eta_{\mathcal{M}} \bigvee \eta_{\mathcal{N}}, \quad \delta_{\mathcal{M}} \bigwedge \delta_{\mathcal{N}})$.

Definition 2.4. Let $\mathcal{A} = (\eta_{\mathcal{A}}, \delta_{\mathcal{A}})$ and $\mathcal{B} = (\eta_{\mathcal{B}}, \delta_{\mathcal{B}})$ be two IF sets in a near ring $\mathcal{N}$. Then we define the *product* $\mathcal{A} \circ \mathcal{B} = (\eta_{\mathcal{A}\circ\mathcal{B}}, \delta_{\mathcal{A}\circ\mathcal{B}})$ in $\mathcal{N}$ as follows:

$$\eta_{\mathcal{A}\circ\mathcal{B}}(n) := \begin{cases} \bigvee_{n=kl} \min\{\eta_{\mathcal{A}}(k), \eta_{\mathcal{B}}(l)\} & \text{if } n = kl \\ 0 & \text{if } n \text{ is not expressible as } n = kl \end{cases}$$

and

$$\delta_{\mathcal{A}\circ\mathcal{B}}(n) := \begin{cases} \bigwedge_{n=ab} \max\{\delta_{\mathcal{A}}(a), \delta_{\mathcal{B}}(b)\} & \text{if } n = ab \\ 1 & \text{if } n \text{ is not expressible as } n = ab. \end{cases}$$

We define the *intrinsic product* of two IF sets in a near ring $\mathcal{N}$ as follows:
Let $\mathcal{A} = (\eta_{\mathcal{A}}, \delta_{\mathcal{A}})$ and $\mathcal{B} = (\eta_{\mathcal{B}}, \delta_{\mathcal{B}})$ be two IF sets in a near ring $\mathcal{N}$. Then intrinsic

product of $\mathcal{A} = (\eta_{\mathcal{A}}, \delta_{\mathcal{A}})$ and $\mathcal{B} = (\eta_{\mathcal{B}}, \delta_{\mathcal{B}})$ is defined to be the IF set $\mathcal{A} * \mathcal{B} = (\eta_{\mathcal{A}*\mathcal{B}}, \delta_{\mathcal{A}*\mathcal{B}})$ in $\mathcal{N}$ given by:

$$\eta_{\mathcal{A}*\mathcal{B}}(n) := \bigvee_{n=\sum\limits_{finite} a_i b_i} \min \begin{cases} \eta_{\mathcal{A}}(a_1), \eta_{\mathcal{A}}(a_2), ..., \eta_{\mathcal{A}}(a_m) \\ \eta_{\mathcal{B}}(b_1), \eta_{\mathcal{B}}(b_2), ..., \eta_{\mathcal{B}}(b_m) \end{cases}$$

and

$$\delta_{\mathcal{A}*\mathcal{B}}(n) := \bigwedge_{n=\sum\limits_{finite} a_i b_i} \max \begin{cases} \delta_{\mathcal{A}}(a_1), \delta_{\mathcal{A}}(a_2), ..., \delta_{\mathcal{A}}(a_m) \\ \delta_{\mathcal{B}}(b_1), \delta_{\mathcal{B}}(b_2), ..., \delta_{\mathcal{B}}(b_m) \end{cases}$$

if $n = \sum\limits_{i=1}^{m} a_i b_i$ for some $a_i, b_i \in \mathcal{N}$ and $m \in \mathbb{Z}^+$ where each $a_i b_i \neq 0$. Otherwise $\mathcal{A} * \mathcal{B} = 0_{\sim}$, i.e., $\eta_{\mathcal{A}*\mathcal{B}}(x) = 0$ and $\delta_{\mathcal{A}*\mathcal{B}}(x) = 1$.

Lemma 2.5. *Let $\mathcal{K}$, $\mathcal{L}$ and $\mathcal{M}$ be IF ideals of a near ring $\mathcal{N}$. Then*

(1) *$\mathcal{K} + \mathcal{L}$ is an IF ideal,*
(2) *$\mathcal{K} * (\mathcal{L} * \mathcal{M}) = (\mathcal{K} * \mathcal{L}) * \mathcal{M}$*

Proof. (1) Let $\mathcal{K} = (\eta_{\mathcal{K}}, \delta_{\mathcal{K}})$ and $\mathcal{L} = (\eta_{\mathcal{L}}, \delta_{\mathcal{L}})$ be two IF ideals of $\mathcal{N}$. It means that $\eta_{\mathcal{K}}, \eta_{\mathcal{L}}$ are fuzzy ideals and $\delta_{\mathcal{K}}, \delta_{\mathcal{L}}$ are anti fuzzy ideals. We know that $\mathcal{K} + \mathcal{L}$ is defined as follows:

$$\mathcal{K} + \mathcal{L} = (\eta_{\mathcal{K}} + \eta_{\mathcal{L}} - \eta_{\mathcal{K}}\eta_{\mathcal{L}}, \delta_{\mathcal{K}}\delta_{\mathcal{L}}).$$

If we show that $\eta_{\mathcal{K}} + \eta_{\mathcal{L}} - \eta_{\mathcal{K}}\eta_{\mathcal{L}}$ is a fuzzy ideal and $\delta_{\mathcal{K}}\delta_{\mathcal{L}}$ is an anti fuzzy ideal, then we are done. Since $\eta_{\mathcal{K}}$ and $\eta_{\mathcal{L}}$ are fuzzy ideals, for any $\mathfrak{u}, \mathfrak{v} \in \mathcal{N}$, we have

$$\begin{aligned} &\eta_{\mathcal{K}+\mathcal{L}}(\mathfrak{u} - \mathfrak{v}) \\ &= \eta_{\mathcal{K}}(\mathfrak{u} - \mathfrak{v}) + \eta_{\mathcal{L}}(\mathfrak{u} - \mathfrak{v}) - \eta_{\mathcal{K}}(\mathfrak{u} - \mathfrak{v}) \cdot \eta_{\mathcal{L}}(\mathfrak{u} - \mathfrak{v}) \\ &\geq \min\{\eta_{\mathcal{K}}(\mathfrak{u}), \eta_{\mathcal{K}}(\mathfrak{v})\} + \min\{\eta_{\mathcal{L}}(\mathfrak{u}), \eta_{\mathcal{L}}(\mathfrak{v})\} - \min\{\eta_{\mathcal{K}}(\mathfrak{u}), \eta_{\mathcal{K}}(\mathfrak{v})\} \cdot \min\{\eta_{\mathcal{L}}(\mathfrak{u}), \eta_{\mathcal{L}}(\mathfrak{v})\} \\ &\geq \min\{\eta_{\mathcal{K}}(\mathfrak{u}) + \eta_{\mathcal{L}}(\mathfrak{u}), \eta_{\mathcal{K}}(\mathfrak{v}) + \eta_{\mathcal{L}}(\mathfrak{v})\} - \min\{\eta_{\mathcal{K}}(\mathfrak{u}) \cdot \eta_{\mathcal{L}}(\mathfrak{u}), \eta_{\mathcal{K}}(\mathfrak{v}) \cdot \eta_{\mathcal{L}}(\mathfrak{v})\} \\ &\geq \min\{\eta_{\mathcal{K}}(\mathfrak{u}) + \eta_{\mathcal{L}}(\mathfrak{u}) - \eta_{\mathcal{K}}(\mathfrak{u}) \cdot \eta_{\mathcal{L}}(\mathfrak{u}), \eta_{\mathcal{K}}(\mathfrak{v}) + \eta_{\mathcal{L}}(\mathfrak{v}) - \eta_{\mathcal{K}}(\mathfrak{v}) \cdot \eta_{\mathcal{L}}(\mathfrak{v})\} \\ &\geq \min\{\eta_{\mathcal{K}+\mathcal{L}}(\mathfrak{u}), \eta_{\mathcal{K}+\mathcal{L}}(\mathfrak{v})\}, \text{ i.e.,} \end{aligned}$$

$$\eta_{\mathcal{K}+\mathcal{L}}(\mathfrak{u} - \mathfrak{v}) \geq \min\{\eta_{\mathcal{K}+\mathcal{L}}(\mathfrak{u}), \eta_{\mathcal{K}+\mathcal{L}}(\mathfrak{v})\}, \tag{2.1}$$

$$\begin{aligned} \eta_{\mathcal{K}+\mathcal{L}}(\mathfrak{v} + \mathfrak{u} - \mathfrak{v}) &= \eta_{\mathcal{K}}(\mathfrak{v} + \mathfrak{u} - \mathfrak{v}) + \eta_{\mathcal{L}}(\mathfrak{v} + \mathfrak{u} - \mathfrak{v}) - \eta_{\mathcal{K}}(\mathfrak{v} + \mathfrak{u} - \mathfrak{v}) \cdot \eta_{\mathcal{L}}(\mathfrak{v} + \mathfrak{u} - \mathfrak{v}) \\ &\geq \eta_{\mathcal{K}}(\mathfrak{u}) + \eta_{\mathcal{L}}(\mathfrak{u}) - \eta_{\mathcal{K}}(\mathfrak{u}) \cdot \eta_{\mathcal{L}}(\mathfrak{u}) \\ &\geq \eta_{\mathcal{K}+\mathcal{L}}(\mathfrak{u}), i.e., \end{aligned}$$

$$\eta_{\mathcal{K}+\mathcal{L}}(\mathfrak{v} + \mathfrak{u} - \mathfrak{v}) \geq \eta_{\mathcal{K}+\mathcal{L}}(\mathfrak{u}), \tag{2.2}$$

$$\begin{aligned} \eta_{\mathcal{K}+\mathcal{L}}(\mathfrak{u}\mathfrak{v}) &= \eta_{\mathcal{K}}(\mathfrak{u}\mathfrak{v}) + \eta_{\mathcal{L}}(\mathfrak{u}\mathfrak{v}) - \eta_{\mathcal{K}}(\mathfrak{u}\mathfrak{v}) \cdot \eta_{\mathcal{L}}(\mathfrak{u}\mathfrak{v}) \\ &\geq \eta_{\mathcal{K}}(\mathfrak{v}) + \eta_{\mathcal{L}}(\mathfrak{v}) - \eta_{\mathcal{K}}(\mathfrak{v}) \cdot \eta_{\mathcal{L}}(\mathfrak{v}) \\ &= \eta_{\mathcal{K}+\mathcal{L}}(\mathfrak{v}), i.e., \end{aligned}$$

$$\eta_{\mathcal{K}+\mathcal{L}}(\mathfrak{u}\mathfrak{v}) \geq \eta_{\mathcal{K}+\mathcal{L}}(\mathfrak{v}). \tag{2.3}$$

Also for any $\mathfrak{u}, \mathfrak{v}, \mathfrak{a} \in \mathcal{N}$,

$$\begin{aligned}\eta_{\mathcal{K}+\mathcal{L}}((\mathfrak{u}+\mathfrak{a})\mathfrak{v}-\mathfrak{u}\mathfrak{v}) &= \eta_{\mathcal{K}}((\mathfrak{u}+\mathfrak{a})\mathfrak{v}-\mathfrak{u}\mathfrak{v}) + \eta_{\mathcal{L}}((\mathfrak{u}+\mathfrak{a})\mathfrak{v}-\mathfrak{u}\mathfrak{v}) - \eta_{\mathcal{K}}((\mathfrak{u}+\mathfrak{a})\mathfrak{v}-\mathfrak{u}\mathfrak{v}) \\ &\quad \cdot \eta_{\mathcal{L}}((\mathfrak{u}+\mathfrak{a})\mathfrak{v}-\mathfrak{u}\mathfrak{v}) \\ &\geq \eta_{\mathcal{K}}(\mathfrak{a}) + \eta_{\mathcal{L}}(\mathfrak{a}) - \eta_{\mathcal{K}}(\mathfrak{a}) \cdot \eta_{\mathcal{L}}(\mathfrak{a}) \\ &= \eta_{\mathcal{K}+\mathcal{L}}(\mathfrak{a}), i.e.,\end{aligned}$$

$$\eta_{\mathcal{K}+\mathcal{L}}((\mathfrak{u}+\mathfrak{a})\mathfrak{v}-\mathfrak{u}\mathfrak{v}) \geq \eta_{\mathcal{K}+\mathcal{L}}(\mathfrak{a}). \tag{2.4}$$

Moreover, we will show that $\delta_{\mathcal{K}+\mathcal{L}}$ is anti fuzzy ideal of $\mathcal{N}$. Since $\delta_{\mathcal{K}}$ and $\delta_{\mathcal{L}}$ are anti fuzzy ideals of $\mathcal{N}$, we get

$$\begin{aligned}\delta_{\mathcal{K}+\mathcal{L}}(\mathfrak{u}-\mathfrak{v}) &= \delta_{\mathcal{K}}(\mathfrak{u}-\mathfrak{v}) \cdot \delta_{\mathcal{L}}(\mathfrak{u}-\mathfrak{v}) \\ &\leq \max\{\delta_{\mathcal{K}}(\mathfrak{u}), \eta_{\mathcal{K}}(\mathfrak{v})\} \cdot \max\{\delta_{\mathcal{L}}(\mathfrak{u}), \delta_{\mathcal{L}}(\mathfrak{v})\} \\ &\leq \max\{\delta_{\mathcal{K}}(\mathfrak{u}) \cdot \delta_{\mathcal{L}}(\mathfrak{u}), \delta_{\mathcal{K}}(\mathfrak{v}) \cdot \eta_{\mathcal{L}}(\mathfrak{v})\} \\ &= \max\{\delta_{\mathcal{K}+\mathcal{L}}(\mathfrak{u}), \delta_{\mathcal{K}+\mathcal{L}}(\mathfrak{v})\}, i.e.,\end{aligned}$$

$$\delta_{\mathcal{K}+\mathcal{L}}(\mathfrak{u}-\mathfrak{v}) \leq \max\{\delta_{\mathcal{K}+\mathcal{L}}(\mathfrak{u}), \delta_{\mathcal{K}+\mathcal{L}}(\mathfrak{v})\} \tag{2.5}$$

and

$$\begin{aligned}\delta_{\mathcal{K}+\mathcal{L}}(\mathfrak{v}+\mathfrak{u}-\mathfrak{v}) &= \delta_{\mathcal{K}}(\mathfrak{v}+\mathfrak{u}-\mathfrak{v}) \cdot \delta_{\mathcal{L}}(\mathfrak{v}+\mathfrak{u}-\mathfrak{v}) \\ &\leq \delta_{\mathcal{K}}(\mathfrak{u}) \cdot \delta_{\mathcal{L}}(\mathfrak{u}) \\ &= \delta_{\mathcal{K}+\mathcal{L}}(\mathfrak{u}), i.e.,\end{aligned}$$

$$\delta_{\mathcal{K}+\mathcal{L}}(\mathfrak{v}+\mathfrak{u}-\mathfrak{v}) \leq \delta_{\mathcal{K}+\mathcal{L}}(\mathfrak{u}). \tag{2.6}$$

Also,

$$\begin{aligned}\delta_{\mathcal{K}+\mathcal{L}}(\mathfrak{u}\mathfrak{v}) &= \delta_{\mathcal{K}}(\mathfrak{u}\mathfrak{v}) \cdot \delta_{\mathcal{L}}(\mathfrak{u}\mathfrak{v}) \\ &\leq \delta_{\mathcal{K}}(\mathfrak{v}) \cdot \delta_{\mathcal{L}}(\mathfrak{v}) \\ &= \delta_{\mathcal{K}+\mathcal{L}}(\mathfrak{v}), i.e.,\end{aligned}$$

$$\delta_{\mathcal{K}+\mathcal{L}}(\mathfrak{u}\mathfrak{v}) \leq \eta_{\mathcal{K}+\mathcal{L}}(\mathfrak{v}). \tag{2.7}$$

Furthermore for any $\mathfrak{u}, \mathfrak{v}, \mathfrak{a} \in \mathcal{N}$, we have

$$\begin{aligned}\delta_{\mathcal{K}+\mathcal{L}}((\mathfrak{u}+\mathfrak{a})\mathfrak{v}-\mathfrak{u}\mathfrak{v}) &= \delta_{\mathcal{K}}((\mathfrak{u}+\mathfrak{a})\mathfrak{v}-\mathfrak{u}\mathfrak{v}) \cdot \eta_{\mathcal{L}}((\mathfrak{u}+\mathfrak{a})\mathfrak{v}-\mathfrak{u}\mathfrak{v}) \\ &\leq \delta_{\mathcal{K}}(\mathfrak{a}) \cdot \delta_{\mathcal{L}}(\mathfrak{a}) \\ &= \delta_{\mathcal{K}+\mathcal{L}}(\mathfrak{a}), i.e.,\end{aligned}$$

$$\delta_{\mathcal{K}+\mathcal{L}}((\mathfrak{u}+\mathfrak{a})\mathfrak{v}-\mathfrak{u}\mathfrak{v}) \leq \delta_{\mathcal{K}+\mathcal{L}}(\mathfrak{a}). \tag{2.8}$$

Equations (2.1)–(2.4) show that $\eta_{\mathcal{K}+\mathcal{L}}$ is a fuzzy ideal and equations (2.5)–(2.8) show that $\delta_{\mathcal{K}+\mathcal{L}}$ is an anti fuzzy ideal of $\mathcal{N}$. Then $\mathcal{K}+\mathcal{L}$ is an IF ideal of $\mathcal{N}$.

(2) The proof runs on the same parallel lines as of proposition 3.3 in [7]. □

Definition 2.6 ([12]). A map $\phi : \mathcal{G} \times \mathfrak{Z} \to \mathfrak{Z}$ defined by $\phi(\mathfrak{g}, \mathfrak{z}) = \mathfrak{g} * \mathfrak{z}$, is said to be an *action* of group $\mathcal{G}$ on a set $\mathcal{Z}$, if for all $\mathfrak{g}, \mathfrak{h} \in \mathcal{G}$, $\mathfrak{z} \in \mathfrak{Z}$,
(i) $\mathfrak{g} * (\mathfrak{h} * \mathfrak{z}) = (\mathfrak{g}\mathfrak{h}) * \mathfrak{z}$,
(ii) $\mathfrak{e} * \mathfrak{z} = \mathfrak{z}$, where $\mathfrak{e}$ is an identity of $\mathcal{G}$.

Definition 2.7 ([12]). If $\mathcal{G}$ is a group acts on a set $\mathcal{X}$, then the set

$$\mathcal{G}x = \{ax/a \in \mathcal{G}\}$$

for $x \in \mathcal{X}$, is said to be an *orbit* of $\mathcal{X}$ in $\mathcal{G}$.

We assume that $\mathcal{G}$ is a finite group acting on $\mathcal{N}$ and define the group action on an IF set of $\mathcal{N}$ as follows:

Definition 2.8. The group action of $\mathcal{G}$ on an IF set $\mathcal{A}$ of $\mathcal{N}$ is given by

$$\mathcal{A}^g = \{\langle x, \eta_{\mathcal{A}}(x^g), \delta_{\mathcal{A}}(x^g)\rangle | \ \ x \in \mathcal{N}, g \in \mathcal{G}\}$$

where x^g means g acts on x. For our simplicity, we write $\mathcal{A}^g$ as $\mathcal{A}^g = (\eta_{\mathcal{A}^g}, \delta_{\mathcal{A}^g})$ if $\mathcal{A} = (\eta_{\mathcal{A}}, \delta_{\mathcal{A}})$ where $\eta_{\mathcal{A}^g}(x) = \eta_{\mathcal{A}}(x^g)$ and $\delta_{\mathcal{A}^g}(x) = \delta_{\mathcal{A}}(x^g)$.

Example 2.9. $\mathcal{N} = \{\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3\}$ is a near ring under following binary operations:

$+$	$\mathfrak{m}_1$	$\mathfrak{m}_2$	$\mathfrak{m}_3$
$\mathfrak{m}_1$	$\mathfrak{m}_1$	$\mathfrak{m}_2$	$\mathfrak{m}_3$
$\mathfrak{m}_2$	$\mathfrak{m}_2$	$\mathfrak{m}_3$	$\mathfrak{m}_1$
$\mathfrak{m}_3$	$\mathfrak{m}_3$	$\mathfrak{m}_1$	$\mathfrak{m}_2$

$\cdot$	$\mathfrak{m}_1$	$\mathfrak{m}_2$	$\mathfrak{m}_3$
$\mathfrak{m}_1$	$\mathfrak{m}_1$	$\mathfrak{m}_1$	$\mathfrak{m}_1$
$\mathfrak{m}_2$	$\mathfrak{m}_1$	$\mathfrak{m}_2$	$\mathfrak{m}_3$
$\mathfrak{m}_3$	$\mathfrak{m}_1$	$\mathfrak{m}_2$	$\mathfrak{m}_3$

and $\mathcal{G} = Aut(\mathcal{N}) = \{\mathfrak{I}, \mathfrak{f}\}$, where $\mathfrak{I}$ is an identity automorphism and automorphism $\mathfrak{f}$ is defined by

$$\mathfrak{f}(\mathfrak{m}_1) = \mathfrak{m}_1, \mathfrak{f}(\mathfrak{m}_2) = \mathfrak{m}_3 \ \text{ and } \ \mathfrak{f}(\mathfrak{m}_3) = \mathfrak{m}_2.$$

$\zeta, \delta : \mathcal{N} \to [0, 1]$ defined by

$$\zeta(\mathfrak{m}) = \begin{cases} 0.7 & \mathfrak{m} = \mathfrak{m}_1 \\ 0.6 & \mathfrak{m} = \mathfrak{m}_2, \mathfrak{m}_3, \end{cases} \quad \delta(\mathfrak{m}) = \begin{cases} 0.3 & \mathfrak{m} = \mathfrak{m}_1 \\ 0.4 & \mathfrak{m} = \mathfrak{m}_2, \mathfrak{m}_3. \end{cases}$$

ζ and δ are fuzzy ideal and anti fuzzy ideal of $\mathcal{N}$ respectively, then $\mathcal{F} = (\zeta, \delta)$ is an IF ideal. Group action on fuzzy sets $\zeta^g, \delta^g : \mathcal{N} \to [0, 1]$ is defined as $\zeta^g(\mathfrak{m}) = \zeta(\mathfrak{m}^g)$ and $\delta^g(\mathfrak{m}) = \delta(\mathfrak{m}^g)$ respectively. Under this action, we can easily see that $\mathcal{A}^g = (\zeta^g, \delta^g)$ is an IF ideal.

3. IF prime ideals

Definition 3.1. Let $\mathcal{P} = (\eta_{\mathcal{P}}, \delta_{\mathcal{P}})$ be an IF ideal of a near ring $\mathcal{N}$. Then $\mathcal{P}$ is called an *IF prime ideal*, if $\eta_{\mathcal{P}}, \delta_{\mathcal{P}}$ are not constant maps and for any two IF ideals $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{N}$, $\mathcal{A} * \mathcal{B} \subseteq \mathcal{P}$ implies that either $\mathcal{A} \subseteq \mathcal{P}$ or $\mathcal{B} \subseteq \mathcal{P}$.

Example 3.2. Consider $\mathcal{N} = \{0, 1, 2, 3\}$ a near ring under binary operations addition modulo 4 and for any $a_1, a_2 \in \mathcal{N}$ multiplication is defined as

$$a_1 \cdot a_2 = \begin{cases} a_2 & a_1 \neq 0 \\ 0 & a_1 = 0. \end{cases}$$

$\mathcal{A} = (\eta_1, \delta_1)$ and $\mathcal{B} = (\eta_2, \delta_2)$ are IF ideals, where fuzzy sets $\eta_1, \eta_2, \delta_1, \delta_2 : Z_4\mathcal{N} \to [0, 1]$ are defined by: for all $a \in \mathcal{N}$, $\eta_1(a) = \begin{cases} 0.9 & a = 0 \\ 0.8 & a \neq 0, \end{cases}$ $\eta_2(a) = 0.9$, $\delta_1(a) = 0.1$ and $\delta_2(a) = \begin{cases} 0 & a = 0 \\ 0.1 & a \neq 0. \end{cases}$ Here, $\mathcal{B}$ is an IF prime ideal of $\mathcal{N}$.

Proposition 3.3. *Let $\mathcal{Q} = (\eta_{\mathcal{Q}}, \delta_{\mathcal{Q}})$ be an IF ideal of $\mathcal{N}$. Then $\mathcal{Q}^g = (\eta_{\mathcal{Q}^g}, \delta_{\mathcal{Q}^g})$ is also an IF ideal.*

Proof. From [Proposition 2 in [9]], $\eta_{\mathcal{Q}^g}$ is a fuzzy ideal of $\mathcal{N}$. Then it remains to prove that $\delta_{\mathcal{Q}^g}$ is an anti fuzzy ideal of $\mathcal{N}$. Since $\delta_{\mathcal{Q}}$ is an anti fuzzy ideal of $\mathcal{N}$, for any $x, y \in \mathcal{N}$, we have

$$\delta_{\mathcal{Q}^g}(x - y) = \delta_{\mathcal{Q}}(x - y)^g = \delta_{\mathcal{Q}}(x^g - y^g) \leq \max(\delta_{\mathcal{Q}}(x^g), \delta_{\mathcal{Q}}(y^g)), i.e.,$$

$$\delta_{\mathcal{Q}^g}(x - y) \leq \max(\delta_{\mathcal{Q}^g}(x), \delta_{\mathcal{Q}^g}(y))) \tag{3.1}$$

and

$$\delta_{\mathcal{Q}^g}(xy) = \delta_{\mathcal{Q}}(xy)^g = \delta_{\mathcal{Q}}(x^g y^g) \leq \max(\delta_{\mathcal{Q}}(x^g), \delta_{\mathcal{Q}}(y^g)), i.e.,$$

$$\delta_{\mathcal{Q}^g}(xy) \leq \max(\delta_{\mathcal{Q}}(x^g), \delta_{\mathcal{Q}}(y^g)). \tag{3.2}$$

Equation (3.1) and (3.2) imply that $\delta_{\mathcal{Q}^g}$ is an anti fuzzy subnear ring of $\mathcal{N}$.
Again for $x, y \in \mathcal{N}$,

$$\delta_{\mathcal{Q}^g}(y + x - y) = \delta_{\mathcal{Q}}(y^g + x^g - y^g) \leq \delta_{\mathcal{Q}}(x)^g = \delta_{\mathcal{Q}^g}(x). \tag{3.3}$$

For $x, y \in \mathcal{N}$, we have

$$\delta_{\mathcal{Q}^g}(xy) = \delta_{\mathcal{Q}}(xy)^g = \delta_{\mathcal{Q}}(x^g y^g) \leq \delta_{\mathcal{Q}}(y^g). \tag{3.4}$$

This implies that $\delta_{\mathcal{Q}^g}$ is an anti fuzzy left ideal of $\mathcal{N}$.
Now, for x, y and $i \in \mathcal{N}$, we have

$$\delta_{\mathcal{Q}^g}((x + i)y - xy) = \delta_{\mathcal{Q}}((x^g + i^g)y^g - x^g y^g) \leq \delta_{\mathcal{Q}}(i^g). \tag{3.5}$$

Thus $\mathcal{Q}^g = (\eta_{\mathcal{Q}^g}, \delta_{\mathcal{Q}^g})$ is an IF ideal of $\mathcal{N}$. □

Proposition 3.4. *Let $\mathcal{Q} = (\eta_{\mathcal{Q}}, \delta_{\mathcal{Q}})$ be an IF prime ideal of $\mathcal{N}$. Then $\mathcal{Q}^g$ is also an IF prime ideal.*

Proof. Let $\mathcal{A} = (\eta_{\mathcal{A}}, \delta_{\mathcal{A}})$ and $\mathcal{B} = (\eta_{\mathcal{B}}, \delta_{\mathcal{B}})$ be two IF ideals of $\mathcal{N}$ such that $\mathcal{A} * \mathcal{B} \subseteq \mathcal{Q}^g$. By proposition 3.3, $\mathcal{A}^{g^{-1}}$ and $\mathcal{B}^{g^{-1}}$ are also IF ideals of $\mathcal{N}$. Now, we claim that $\mathcal{A}^{g^{-1}} * \mathcal{B}^{g^{-1}} \subseteq Q$, i.e., for $\mathcal{A}^{g^{-1}} = (\eta_{\mathcal{A}^{g^{-1}}}, \delta_{\mathcal{A}^{g^{-1}}})$ and $\mathcal{B}^{g^{-1}} = (\eta_{\mathcal{B}^{g^{-1}}}, \delta_{\mathcal{B}^{g^{-1}}})$,

$\eta_{\mathcal{A}^{g^{-1}}*\mathcal{B}^{g^{-1}}} \subseteq \eta_{\mathcal{Q}}$ and $\delta_{\mathcal{A}^{g^{-1}}*\mathcal{B}^{g^{-1}}} \supseteq \delta_{\mathcal{Q}}$. For every $n \in \mathcal{N}, \quad m \in \mathbb{Z}^+$

$$\begin{aligned}
\eta_{\mathcal{A}^{g^{-1}}*\mathcal{B}^{g^{-1}}}(n) &= \bigvee_{n=\sum\limits_{finite} a_i b_i} \min \begin{cases} \eta_{\mathcal{A}^{g^{-1}}}(a_1), \eta_{\mathcal{A}^{g^{-1}}}(a_2), \cdots, \eta_{\mathcal{A}^{g^{-1}}}(a_m) \\ \eta_{\mathcal{B}^{g^{-1}}}(b_1), \eta_{\mathcal{B}^{g^{-1}}}(b_2), \cdots, \eta_{\mathcal{B}^{g^{-1}}}(b_m) \end{cases} \\
&= \bigvee_{n^{g^{-1}}=\sum\limits_{finite} a_i^{g^{-1}} b_i^{g^{-1}}} \min \begin{cases} \eta_{\mathcal{A}}(a_1^{g^{-1}}), \eta_{\mathcal{A}}(a_2^{g^{-1}}), \cdots, \eta_{\mathcal{A}}(a_m^{g^{-1}}) \\ \eta_{\mathcal{B}}(b_1^{g^{-1}}), \eta_{\mathcal{B}}(b_2^{g^{-1}}), \cdots, \eta_{\mathcal{B}}(b_m^{g^{-1}}) \end{cases} \\
&= \eta_{\mathcal{A}*\mathcal{B}}(n^{g^{-1}}) \leq \eta_{\mathcal{Q}^g}(n^{g^{-1}}) = \eta_{\mathcal{Q}}(n), i.e.,
\end{aligned}$$

$$\eta_{\mathcal{A}^{g^{-1}}*\mathcal{B}^{g^{-1}}}(n) \subseteq \eta_{\mathcal{Q}}$$

and

$$\begin{aligned}
\delta_{\mathcal{A}^{g^{-1}}*\mathcal{B}^{g^{-1}}}(n) &= \bigwedge_{n=\sum\limits_{finite} a_i b_i} \max \begin{cases} \delta_{\mathcal{A}^{g^{-1}}}(a_1), \delta_{\mathcal{A}^{g^{-1}}}(a_2), \cdots, \delta_{\mathcal{A}^{g^{-1}}}(a_m) \\ \delta_{\mathcal{B}^{g^{-1}}}(b_1), \delta_{\mathcal{B}^{g^{-1}}}(b_2), \cdots, \delta_{\mathcal{B}^{g^{-1}}}(b_m) \end{cases} \\
&= \bigwedge_{n^{g^{-1}}=\sum\limits_{finite} a_i^{g^{-1}} b_i^{g^{-1}}} \max \begin{cases} \delta_{\mathcal{A}}(a_1^{g^{-1}}), \delta_{\mathcal{A}}(a_2^{g^{-1}}), \cdots, \delta_{\mathcal{A}}(a_m^{g^{-1}}) \\ \delta_{\mathcal{B}}(b_1^{g^{-1}}), \delta_{\mathcal{B}}(b_2^{g^{-1}}), \cdots, \delta_{\mathcal{B}}(b_m^{g^{-1}}) \end{cases} \\
&= \delta_{\mathcal{A}*\mathcal{B}}(n^{g^{-1}}) \geq \delta_{\mathcal{Q}^g}(n^{g^{-1}}) = \delta_{\mathcal{Q}}(n), i.e.,
\end{aligned}$$

$$\delta_{\mathcal{A}^{g^{-1}}*\mathcal{B}^{g^{-1}}}(n) \supseteq \delta_{\mathcal{Q}}.$$

This implies that $\mathcal{A}^{g^{-1}} * \mathcal{B}^{g^{-1}} \subseteq \mathcal{Q}$. Since $\mathcal{Q}$ is an IF prime ideal, either $\mathcal{A}^{g^{-1}} \subseteq \mathcal{Q}$ or $\mathcal{B}^{g^{-1}} \subseteq \mathcal{Q}$. Thus $\mathcal{A} \subseteq \mathcal{Q}^g$ or $\mathcal{B} \subseteq \mathcal{Q}^g$. □

Following the definition of a $\mathcal{G}$-invariant IF ideal of a ring [10], we define a $\mathcal{G}$-invariant IF ideal of a near ring.

Definition 3.5. Let $\mathcal{A} = (\eta_{\mathcal{A}}, \delta_{\mathcal{A}})$ be an IF ideal of $\mathcal{N}$. Then $\mathcal{A}$ is said to be a *$\mathcal{G}$-invariant IF ideal* of $\mathcal{N}$, if $\eta_{\mathcal{A}^g}(n) = \eta_{\mathcal{A}}(n^g) \geq \eta_{\mathcal{A}}(n)$ and $\delta_{\mathcal{A}^g}(n) = \delta_{\mathcal{A}}(n^g) \leq \delta_{\mathcal{A}}(n)$ for all $g \in G$ and $n \in \mathcal{N}$.

Example 3.6. Consider $\mathcal{N} = \{\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3, \mathfrak{m}_4\}$ a near ring under following binary operations:

$+$	$\mathfrak{m}_1$	$\mathfrak{m}_2$	$\mathfrak{m}_3$	$\mathfrak{m}_4$
$\mathfrak{m}_1$	$\mathfrak{m}_1$	$\mathfrak{m}_2$	$\mathfrak{m}_3$	$\mathfrak{m}_4$
$\mathfrak{m}_2$	$\mathfrak{m}_2$	$\mathfrak{m}_1$	$\mathfrak{m}_4$	$\mathfrak{m}_3$
$\mathfrak{m}_3$	$\mathfrak{m}_3$	$\mathfrak{m}_4$	$\mathfrak{m}_2$	$\mathfrak{m}_1$
$\mathfrak{m}_4$	$\mathfrak{m}_4$	$\mathfrak{m}_3$	$\mathfrak{m}_1$	$\mathfrak{m}_2$

$\cdot$	$\mathfrak{m}_1$	$\mathfrak{m}_2$	$\mathfrak{m}_3$	$\mathfrak{m}_4$
$\mathfrak{m}_1$	$\mathfrak{m}_1$	$\mathfrak{m}_1$	$\mathfrak{m}_1$	$\mathfrak{m}_1$
$\mathfrak{m}_2$	$\mathfrak{m}_1$	$\mathfrak{m}_1$	$\mathfrak{m}_1$	$\mathfrak{m}_1$
$\mathfrak{m}_3$	$\mathfrak{m}_1$	$\mathfrak{m}_1$	$\mathfrak{m}_1$	$\mathfrak{m}_1$
$\mathfrak{m}_4$	$\mathfrak{m}_1$	$\mathfrak{m}_1$	$\mathfrak{m}_2$	$\mathfrak{m}_2$

and $\mathcal{G} = Aut(\mathcal{N})$. Then IF set $\mathcal{A} = (\eta_{\mathcal{A}}, \delta_{\mathcal{A}})$ in $\mathcal{N}$ defined by $\eta_{\mathcal{A}}(\mathfrak{m}_1) = 0.8, \eta_{\mathcal{A}}(\mathfrak{m}_2) = 0.6, \eta_{\mathcal{A}}(\mathfrak{m}_3) = \eta_{\mathcal{A}}(\mathfrak{m}_4) = 0.3$ and $\delta_{\mathcal{A}}(\mathfrak{m}_1) = 0.2, \delta_{\mathcal{A}}(\mathfrak{m}_2) = 0.3, \delta_{\mathcal{A}}(\mathfrak{m}_3) = \delta_{\mathcal{A}}(\mathfrak{m}_4) = 0.7$, is an IF ideal which is $\mathcal{G}$-invariant.

Theorem 3.7. *Let $\mathcal{F}$ be an IF ideal of $\mathcal{N}$ and $\mathcal{F}^{\mathcal{G}} = \bigcap\limits_{g \in \mathcal{G}} \mathcal{F}^g$. Then $\mathcal{F}^{\mathcal{G}}$ is the largest $\mathcal{G}$-invariant IF ideal of $\mathcal{N}$ contained in $\mathcal{F}$.*

Proof. It is clear that $\mathcal{F}^{\mathcal{G}}$ is an ideal of $\mathcal{N}$, because

$$\mathcal{F}^{\mathcal{G}}(n) = \underset{g\in\mathcal{G}}{\cap}\mathcal{F}^{g}(n) = (\eta_{\mathcal{F}^G}(n), \delta_{\mathcal{F}^G}(n))$$
$$= (\min_{g\in\mathcal{G}}(\eta_{\mathcal{F}}(n^g)), \max_{g\in\mathcal{G}}(\delta_{\mathcal{F}}(n^g))).$$

We know that an IF ideal $\mathcal{F}$ of $\mathcal{N}$ is $\mathcal{G}$-invariant if and only if $\mathcal{F} = \mathcal{F}^{\mathcal{G}}$. Assume that $\mathcal{G}$-invariant IF ideal $\mathcal{E}$ contained in $\mathcal{F}$. Then for any $g \in \mathcal{G}$ and $n \in \mathcal{N}$, $\eta_{\mathcal{E}}(n^g) = \eta_{\mathcal{E}}(n) \leq \eta_{\mathcal{F}}(n)$. Also $\eta_{\mathcal{E}}(n^g) = \eta_{\mathcal{E}}(n) = \eta_{\mathcal{E}}(n^g)^{g^{-1}} \leq \eta_{\mathcal{F}^g}(n^g)$, we get

$$\eta_{\mathcal{E}} \subseteq \eta_{\mathcal{F}^G}. \tag{3.6}$$

Also, $\delta_{\mathcal{E}}(n^g) = \delta_{\mathcal{E}}(n) \geq \delta_{\mathcal{F}}(n)$. Also $\delta_{\mathcal{E}}(n^g) = \delta_{\mathcal{E}}(n) = \delta_{\mathcal{E}}(n^g)^{g^{-1}} \geq \delta_{\mathcal{F}^g}(n^g)$, we get

$$\delta_{\mathcal{E}} \supseteq \delta_{\mathcal{F}^G}. \tag{3.7}$$

By equations (3.6) and (3.7), $\mathcal{E} \subset \mathcal{F}^{\mathcal{G}}$. This shows that $\mathcal{F}^{\mathcal{G}}$ is the largest $\mathcal{G}$-invariant IF ideal of $\mathcal{N}$. □

The characterization of $\mathcal{G}$-invariant IF ideals can be directly obtained from the definition. This is summarized in the following remark.

Remark 3.8. An IF ideal $\mathcal{F}$ of $\mathcal{N}$ is $\mathcal{G}$-invariant IF ideal of $\mathcal{N}$ if and only if $\mathcal{F} = \mathcal{F}^G$. Clearly follows by definition 3.5.

4. Union of IF ideals

Definition 4.1. If $\{\mathcal{F}_n = (\eta_{\mathcal{F}_n}, \delta_{\mathcal{F}_n})\}$ is nonempty collection of IF sets of a near ring $\mathcal{N}$, then the *union* of IF sets is defined as

$$\bigcup_{i\in I}\mathcal{F}_n = (\eta_{\underset{i\in I}{\bigcup}\mathcal{F}_i}, \delta_{\underset{i\in I}{\bigcup}\mathcal{F}_i})$$

where $\eta_{\underset{i\in I}{\bigcup}\mathcal{F}_i} = \bigvee_{i\in I}\eta_{\mathcal{F}_i}$ and $\delta_{\underset{i\in I}{\bigcup}\mathcal{F}_i} = \bigwedge_{i\in I}\delta_{\mathcal{F}_i}$.

We know that the union of IF ideals of a near ring $\mathcal{N}$ is not an IF ideal of $\mathcal{N}$, in general.

Example 4.2. Let $\mathcal{S}$ be a near ring. Then

$$\mathcal{N} = \left\{ \begin{pmatrix} a_1 & a_2 \\ 0 & 0 \end{pmatrix} \middle| a_1, a_2, 0 \in \mathcal{S} \right\}$$

is a near ring with respect to matrix addition and matrix multiplication. Take

$$\mathcal{J}_1 = \left\{ \begin{pmatrix} 0 & a_2 \\ 0 & 0 \end{pmatrix} \middle| a_2, 0 \in \mathcal{S} \right\} \text{ and } \mathcal{J}_2 = \left\{ \begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix} \middle| a_1, 0 \in \mathcal{S} \right\}$$

ideals of $\mathcal{N}$. Define maps $\mu_1, \mu_2, \delta_1, \delta_2 : \mathcal{N} \to [0,1]$ by $\mu_1(x) = \begin{cases} 0.7 & x \in \mathcal{J}_1 \\ 0 & x \notin \mathcal{J}_1, \end{cases}$ $\mu_2(x) = \begin{cases} 0.5 & x \in \mathcal{J}_2 \\ 0 & x \notin \mathcal{J}_2, \end{cases}$ $\delta_1(x) = \begin{cases} 0 & x \in \mathcal{J}_1 \\ 0.7 & x \notin \mathcal{J}_1, \end{cases}$ and $\delta_2(x) = \begin{cases} 0 & x \in \mathcal{J}_2 \\ 0.5 & x \notin \mathcal{J}_2. \end{cases}$ Then

$\mathcal{A} = (\mu_1, \delta_1)$ and $\mathcal{B} = (\mu_2, \delta_2)$ are IF ideals in $\mathcal{N}$. But $\mathcal{A} \cup \mathcal{B}$ is not an IF ideal in $\mathcal{N}$, since

$$\mu_1 \cup \mu_2(n) = \begin{cases} max\{0.7, 0.5\} & x \in \mathcal{J}_1 \cup \mathcal{J}_2 \\ 0 & x \notin \mathcal{I}_J \cup \mathcal{J}_2 \end{cases}$$

is not a fuzzy ideal of $\mathcal{N}$, for $n = \begin{pmatrix} 0 & a_2 \\ 0 & 0 \end{pmatrix}$ and $m = \begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix}$, $n - m = \begin{pmatrix} -a_1 & a_2 \\ 0 & 0 \end{pmatrix} \notin \mathcal{J}_1 \cup \mathcal{J}_2$. We see that $\mu_1 \cup \mu_2(n - m) = 0$, $\mu_1 \cup \mu_2(n) = 0.7$ and $\mu_1 \cup \mu_2(m) = 0.5$. Thus

$$\begin{aligned} \mu_1 \cup \mu_2(n - m) = 0 &\ngtr max\{\mu_1 \cup \mu_2(n), \mu_1 \cup \mu_2(m)\} \\ &\ngtr max\{0.7, 0.5\} \\ &\ngtr 0.7. \end{aligned}$$

So $\mu_1 \cup \mu_2$ is not a fuzzy ideal of $\mathcal{N}$. Hence $\mathcal{A} \cup \mathcal{B}$ not an IF ideal of $\mathcal{N}$.

Lemma 4.3. *Let $P = (\eta_{\mathcal{P}}, \delta_{\mathcal{P}})$ and $\mathcal{Q} = (\eta_{\mathcal{Q}}, \delta_{\mathcal{Q}})$ be two $\mathcal{G}$-invariant IF ideals of $\mathcal{N}$. Then $\mathcal{P} * \mathcal{Q} = (\eta_{\mathcal{P}*\mathcal{Q}}, \delta_{\mathcal{P}*\mathcal{Q}})$ is a $\mathcal{G}$-invariant.*

Proof. The proof runs on the same parrallel lines as that of Lemma 3.7 in [10]. □

Proposition 4.4. *If $\{\mathcal{F}_n\}$ is a chain of IF ideals of $\mathcal{N}$, then for any $x, y \in \mathcal{N}$*

(i) $$\min(\bigvee_n \{\eta_{F_n}(x)\}, \bigvee_n \{\eta_{F_n}(y)\}) = \bigvee_n \{\min(\eta_{F_n}(x), \eta_{F_n}(y))\}.$$

and

$$\max(\bigwedge_n \{\delta_{F_n}(x)\}, \bigwedge_n \{\delta_{F_n}(y)\}) = \bigwedge_n \{\max(\delta_{F_n}(x), \delta_{F_n}(y))\},$$

where $\mathcal{F}_i = (\eta_{\mathcal{F}_i}, \delta_{\mathcal{F}_i})$, $i \in \mathbb{N}$ and $\eta_{\mathcal{F}_i} \leq \eta_{\mathcal{F}_{i+1}}$, $\delta_{\mathcal{F}_i} \geq \delta_{\mathcal{F}_{i+1}}$.

Proof. Assume that $u = \max(\bigwedge_n \{\delta_{F_n}(x)\}, \bigwedge_n \{\delta_{F_n}(y)\}) < \bigwedge_n \{\max(\delta_{F_n}(x), \delta_{F_n}(y))\} = v$. Then $\bigwedge_n \delta_{F_n}(x), \bigwedge_n \delta_{F_n}(y) < v$. Thus there exist s and t such that $\delta_{F_s}(x) < v$ and $\delta_{F_t}(y) < v$. So for some $m(> s, t)$, $\delta_{F_s}(x) \geq \delta_{F_m}(x)$ and $\delta_{F_s}(y) \geq \delta_{F_m}(y)$. Hence $\max(\delta_{F_m}(x), \delta_{F_m}(y)) < v$. Which contradict our assumption $\bigwedge_n \{\max(\delta_{F_n}(x), \delta_{F_n}(y))\} = v$. On the other hand, suppose that $u > v$. Without loss of generality, we can assume that $u = \bigwedge_n (\delta_{F_n}(x))$. Then there exist some m such that $\max(\delta_{F_m}(x), \delta_{F_m}(y)) < u$. This contradiction to the fact $u \leq \delta_{F_m}(x)$. The equality (i) follows by Proposition 3 in [9]. □

Theorem 4.5. *Let $\{\mathcal{F}_n\}$ be a chain of IF ideals of near ring $\mathcal{N}$. Then $\underset{n}{\cup} \mathcal{F}_n$ is an IF ideal of $\mathcal{N}$.*

Proof. Let $\mathcal{A}_n = (\eta_{\mathcal{A}_n}, \delta_{\mathcal{A}_n})$ be an IF ideal, where $\eta_{\mathcal{A}_n}, \delta_{\mathcal{A}_n}$ are fuzzy ideal and anti fuzzy ideal of $\mathcal{N}$ respectively. Then from [Theorem 2 in [9]], $\eta_{\underset{n \in I}{\cup} \mathcal{F}_n}$ is a fuzzy ideal of $\mathcal{N}$. Moreover, by using proposition 4.4, we can easily prove that $\delta_{\underset{n \in I}{\cup} \mathcal{F}_n}$ is an anti fuzzy ideal. Thus $\underset{n}{\cup} \mathcal{F}_n$ is an IF ideal of $\mathcal{N}$. □

5. IF $\mathcal{G}$-prime ideals

Definition 5.1. Let $\mathcal{P}$ be the non constant $\mathcal{G}$-invariant IF ideal of $\mathcal{N}$. Then $\mathcal{P}$ is said to be an *IF $\mathcal{G}$-prime ideal*, if for any two $\mathcal{G}$-invariant IF ideals $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{N}$ such that $\mathcal{A} * \mathcal{B} \subseteq \mathcal{P}$ implies that either $\mathcal{A} \subseteq \mathcal{P}$ or $\mathcal{B} \subseteq \mathcal{P}$.

Example 5.2. Let $\mathcal{M}(\mathcal{Z}_2) = \{f | f : \mathcal{Z}_2 \to \mathcal{Z}_2\}$ be a near ring under pointwise addition and multiplication as composition of functions, i.e., $\mathcal{M}(\mathcal{Z}_2) = \{0, e, f, g\}$, where $0, i$ are zero and identity maps respectively and f, g are defined as $f(0) = 1, f(1) = 0$ and $g(0) = g(1) = 1$. Let IF set $\mathcal{A} = (\eta_{\mathcal{A}}, \delta_{\mathcal{A}})$ defined by $\eta_{\mathcal{A}}(0) = 0.8, \eta_{\mathcal{A}}(e) = 0.6, \eta_{\mathcal{A}}(f) = \eta_{\mathcal{A}}(g) = 0.3$ and $\delta_{\mathcal{A}}(0) = 0.2, \delta_{\mathcal{A}}(e) = 0.3, \delta_{\mathcal{A}}(f) = \delta_{\mathcal{A}}(g) = 0.7$. Then it is easy to show that $\mathcal{A} = (\eta_{\mathcal{A}}, \delta_{\mathcal{A}})$ is $\mathcal{G}$-invariant. IF set $\mathcal{B} = (\eta_{\mathcal{B}}, \delta_{\mathcal{B}})$ in $\mathcal{N}$ defined by $\eta_{\mathcal{B}}(0) = 0.7, \eta_{\mathcal{B}}(e) = 0.5, \eta_{\mathcal{B}}(f) = \eta_{\mathcal{B}}(g) = 0.2$ and $\delta_{\mathcal{B}}(0) = 0.3, \delta_{\mathcal{B}}(e) = 0.4, \delta_{\mathcal{B}}(f) = \delta_{\mathcal{B}}(g) = 0.8$ is also $\mathcal{G}$-invariant. We can see that $\mathcal{A} * \mathcal{B} \subseteq \mathcal{A}$. Thus $\mathcal{A}$ is a $\mathcal{G}$-invariant prime ideal.

Proposition 5.3. *If $\mathcal{N}$ is a zero symmetric near ring and $\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_p$ are IF ideals of $\mathcal{N}$, then*

$$\mathcal{A}_1 * \mathcal{A}_2 * \cdots * \mathcal{A}_p \subseteq \mathcal{A}_1 \cap \mathcal{A}_2 \cap \cdots \cap \mathcal{A}_p.$$

Proof. Let $\mathcal{A}_i = (\eta_{\mathcal{A}_i}, \delta_{\mathcal{A}_i})$ be IF ideal of $\mathcal{N}$. Then intrinsic product $\mathcal{A}_1 * \mathcal{A}_2 = (\eta_{\mathcal{A}_1 * \mathcal{A}_2}, \delta_{\mathcal{A}_1 * \mathcal{A}_2})$ in $\mathcal{N}$ is given by

$$\eta_{\mathcal{A}_1 * \mathcal{A}_2}(n) = \bigvee_{x = \sum_{finite} a_i b_i} \min \begin{cases} \eta_{\mathcal{A}_1}(a_1), \eta_{\mathcal{A}_1}(a_2), \cdots, \eta_{\mathcal{A}_1}(a_m) \\ \eta_{\mathcal{A}_2}(b_1), \eta_{\mathcal{A}_2}(b_2), \cdots, \eta_{\mathcal{A}_2}(b_m), \end{cases}$$

$$\delta_{\mathcal{A}_1 * \mathcal{A}_2}(n) = \bigwedge_{x = \sum_{finite} a_i b_i} \max \begin{cases} \delta_{\mathcal{A}_1}(a_1), \delta_{\mathcal{A}_1}(a_2), \cdots, \delta_{\mathcal{A}_1}(a_m) \\ \delta_{\mathcal{A}_2}(b_1), \delta_{\mathcal{A}_2}(b_2), \cdots, \delta_{\mathcal{A}_2}(b_m) \end{cases}$$

and

$$\mathcal{A}_1 \cap \mathcal{A}_2 = (\min(\eta_{\mathcal{A}_1}, \eta_{\mathcal{A}_2}), \max(\delta_{\mathcal{A}_1}, \delta_{\mathcal{A}_2})).$$

We have to show that $\eta_{\mathcal{A}_1 * \mathcal{A}_2} \subseteq \min(\eta_{\mathcal{A}_1}, \eta_{\mathcal{A}_2})$ and $\delta_{\mathcal{A}_1 * \mathcal{A}_2} \subseteq \max(\delta_{\mathcal{A}_1}, \delta_{\mathcal{A}_2})$. We prove this result by method of mathematical induction.

For $p = 1$, it is obvious.

For $p = 2$, $\mathcal{A}_1 = (\eta_{\mathcal{A}_1}, \delta_{\mathcal{A}_1})$ and $\mathcal{A}_2 = (\eta_{\mathcal{A}_2}, \delta_{\mathcal{A}_2})$. Let n be expressible as $n = a_1 b_1 + a_2 b_2 + \cdots + a_m b_m$, where $a_i, b_i \in \mathcal{N}$ and $a_i b_i \neq 0$. Then

$$\begin{aligned} &\min\{\eta_{\mathcal{A}_1}(a_1), \eta_{\mathcal{A}_1}(a_1), \cdots, \eta_{\mathcal{A}_1}(a_m), \eta_{\mathcal{A}_2}(b_1), \eta_{\mathcal{A}_2}(b_2), \cdots, \eta_{\mathcal{A}_2}(b_m)\} \\ &\leq \min\{\eta_{\mathcal{A}_2}(b_1), \eta_{\mathcal{A}_2}(b_2), \cdots, \eta_{\mathcal{A}_2}(b_m)\} \end{aligned}$$

Since $\eta_{\mathcal{A}_2}$ is fuzzy ideal, we have

$$\begin{aligned} \min\{\eta_{\mathcal{A}_2}(b_1), \eta_{\mathcal{A}_2}(b_2), \cdots, \eta_{\mathcal{A}_2}(b_m)\} &\leq \min\{\eta_{\mathcal{A}_2}(a_1 b_1), \eta_{\mathcal{A}_2}(a_2 b_2), \cdots, \eta_{\mathcal{A}_2}(a_m b_m)\} \\ &\leq \eta_{\mathcal{A}_2}(a_1 b_1 + a_2 b_2 + \cdots + a_m b_m), i.e., \end{aligned}$$

(5.1)

$$\min\{\eta_{\mathcal{A}_1}(a_1), \eta_{\mathcal{A}_2}(a_1), \cdots, \eta_{\mathcal{A}_1}(a_m), \eta_{\mathcal{A}_2}(b_1), \eta_{\mathcal{A}_2}(b_2), \cdots, \eta_{\mathcal{A}_2}(b_m)\} \leq \eta_{\mathcal{A}_2}(n)$$

and

$$\min\{\eta_{\mathcal{A}_1}(a_1),\eta_{\mathcal{A}_1}(a_1),\cdots,\eta_{\mathcal{A}_1}(a_m),\eta_{\mathcal{A}_2}(b_1),\eta_{\mathcal{A}_2}(b_2),\cdots,\eta_{\mathcal{A}_2}(b_m)\}$$
$$\leq \min\{\eta_{\mathcal{A}_1}(a_1),\eta_{\mathcal{A}_1}(a_2),\cdots,\eta_{\mathcal{A}_1}(a_m)\}.$$

Since $\mathcal{N}$ is a zero symmetric near ring and $\eta_{\mathcal{A}_1}$ is a fuzzy ideal, we get

$$\min\{\eta_{\mathcal{A}_1}(a_1),\eta_{\mathcal{A}_1}(a_2),\cdots,\eta_{\mathcal{A}_1}(a_m)\}$$
$$\leq \min\{\eta_{\mathcal{A}_1}((0+a_1)b_1-0b_1),\eta_{\mathcal{A}_1}((0+a_2)b_2-0b_2),\cdots,\eta_{\mathcal{A}_1}((0+a_m)b_m-0b_m)\}$$
$$\leq \eta_{\mathcal{A}_1}(a_1b_1+a_2b_2+\cdots+a_mb_m).$$

Thus

(5.2)
$$\min\{\eta_{\mathcal{A}_1}(a_1),\eta_{\mathcal{A}_2}(a_1),\cdots,\eta_{\mathcal{A}_1}(a_m),\eta_{\mathcal{A}_2}(b_1),\eta_{\mathcal{A}_2}(b_2),\cdots,\eta_{\mathcal{A}_2}(b_m)\}\leq \eta_{\mathcal{A}_1}(n).$$

Furthermore, we get

$$\max\{\delta_{\mathcal{A}_1}(a_1),\delta_{\mathcal{A}_1}(a_1),\cdots,\delta_{\mathcal{A}_1}(a_m),\delta_{\mathcal{A}_2}(b_1),\delta_{\mathcal{A}_2}(b_2),\cdots,\delta_{\mathcal{A}_2}(b_m)\}$$
$$\geq \max\{\delta_{\mathcal{A}_2}(b_1),\delta_{\mathcal{A}_2}(b_2),\cdots,\delta_{\mathcal{A}_2}(b_m)\}$$
$$\geq \max\{\delta_{\mathcal{A}_2}(a_1b_1),\delta_{\mathcal{A}_2}(a_2b_2),\cdots,\eta_{\mathcal{A}_2}(a_mb_m)\}$$
$$\geq \delta_{\mathcal{A}_2}(a_1b_1+a_2b_2+\cdots+a_mb_m),\text{ i.e.,}$$

(5.3)
$$\max\{\delta_{\mathcal{A}_1}(a_1),\delta_{\mathcal{A}_2}(a_1),\cdots,\delta_{\mathcal{A}_1}(a_m),\delta_{\mathcal{A}_2}(b_1),\delta_{\mathcal{A}_2}(b_2),\cdots,\delta_{\mathcal{A}_2}(b_m)\}\geq \delta_{\mathcal{A}_2}(n)$$

and

$$\max\{\delta_{\mathcal{A}_1}(a_1),\delta_{\mathcal{A}_1}(a_1),\cdots,\delta_{\mathcal{A}_1}(a_m),\delta_{\mathcal{A}_2}(b_1),\delta_{\mathcal{A}_2}(b_2),\cdots,\delta_{\mathcal{A}_2}(b_m)\}$$
$$\geq \max\{\delta_{\mathcal{A}_1}(a_1),\delta_{\mathcal{A}_1}(a_2),\cdots,\delta_{\mathcal{A}_1}(a_m)\}$$
$$\geq \max\{\delta_{\mathcal{A}_1}((0+a_1)b_1-0b_1),\delta_{\mathcal{A}_1}((0+a_2)b_2-0b_2),\cdots,\eta_{\mathcal{A}_2}((0+a_m)b_m-0b_m)\}$$
$$\geq \delta_{\mathcal{A}_1}(a_1b_1+a_2b_2+\cdots+a_mb_m),\text{ i.e.,}$$

(5.4)
$$\max\{\delta_{\mathcal{A}_1}(a_1),\delta_{\mathcal{A}_2}(a_1),\cdots,\delta_{\mathcal{A}_1}(a_m),\delta_{\mathcal{A}_2}(b_1),\delta_{\mathcal{A}_2}(b_2),\cdots,\delta_{\mathcal{A}_2}(b_m)\}\geq \delta_{\mathcal{A}_1}(n).$$

It follows from the equation (5.1) that

$$\begin{aligned}\eta_{\mathcal{A}_1*\mathcal{A}_2}(n) &= \bigvee_{x=\sum\limits_{finite} a_ib_i} \min\begin{cases}\eta_{\mathcal{A}_1}(a_1),\eta_{\mathcal{A}_1}(a_2),\cdots,\eta_{\mathcal{A}_1}(a_m)\\ \eta_{\mathcal{A}_2}(b_1),\eta_{\mathcal{A}_2}(b_2),\cdots,\eta_{\mathcal{A}_2}(b_m)\end{cases}\\ &\leq \min\{\eta_{\mathcal{A}_1}(a_1),\eta_{\mathcal{A}_1}(a_2),\cdots,\eta_{\mathcal{A}_1}(a_m),\eta_{\mathcal{A}_2}(b_1),\eta_{\mathcal{A}_2}(b_2),\cdots,\eta_{\mathcal{A}_2}(b_m)\}\\ &\leq \eta_{\mathcal{A}_2}(n), i.e.,\end{aligned}$$

(5.5)
$$\eta_{\mathcal{A}_1*\mathcal{A}_2}(n)\leq \eta_{\mathcal{A}_2}(n)$$

and from the equation (5.3),

$$\begin{aligned}\delta_{\mathcal{A}_1*\mathcal{A}_2}(n) &= \bigwedge_{x=\sum\limits_{finite} a_ib_i} \max\begin{cases}\delta_{\mathcal{A}_1}(a_1),\delta_{\mathcal{A}_1}(a_2),\cdots,\delta_{\mathcal{A}_1}(a_m)\\ \delta_{\mathcal{A}_2}(b_1),\delta_{\mathcal{A}_2}(b_2),\cdots,\delta_{\mathcal{A}_2}(b_m)\end{cases}\\ &\geq \max\{\delta_{\mathcal{A}_1}(a_1),\delta_{\mathcal{A}_1}(a_2),\cdots,\delta_{\mathcal{A}_1}(a_m),\delta_{\mathcal{A}_2}(b_1),\delta_{\mathcal{A}_2}(b_2),\cdots,\delta_{\mathcal{A}_2}(b_m)\}\\ &\geq \delta_{\mathcal{A}_2}(n), i.e.,\end{aligned}$$

$$\delta_{\mathcal{A}_1 * \mathcal{A}_2}(n) \geq \delta_{\mathcal{A}_2}(n). \tag{5.6}$$

Similarly, we can show that

$$\eta_{\mathcal{A}_1 * \mathcal{A}_2}(n) \leq \eta_{\mathcal{A}_1}(n) \tag{5.7}$$

$$\delta_{\mathcal{A}_1 * \mathcal{A}_2}(n) \geq \delta_{\mathcal{A}_1}(n). \tag{5.8}$$

From the equations (5.5), (5.6), (5.7) and (5.8), $\eta_{\mathcal{A}_1 * \mathcal{A}_2}(n) \leq \{\eta_{\mathcal{A}_1}(n), \eta_{\mathcal{A}_2}(n)\}$ and $\delta_{\mathcal{A}_1 * \mathcal{A}_2}(n) \geq \max\{\delta_{\mathcal{A}_1}(n), \delta_{\mathcal{A}_2}(n)\}$. So $\mathcal{A}_1 * \mathcal{A}_2 \subseteq \mathcal{A}_1 \cap \mathcal{A}_2$.

Now assume that it is true for $p = k$. We have to show that it will be true for $k + 1$.

Let $n \in \mathcal{N}$. Then we have

$$\begin{aligned} &\min\{\eta_{\mathcal{A}_1 * \mathcal{A}_2 * \cdots * \mathcal{A}_k}(a_1), \eta_{\mathcal{A}_1 * \mathcal{A}_2 * \cdots * \mathcal{A}_k}(a_2), \cdots, \eta_{\mathcal{A}_1 * \mathcal{A}_2 * \cdots * \mathcal{A}_k}(a_m), \\ &\qquad \eta_{\mathcal{A}_{k+1}}(b_1), \eta_{\mathcal{A}_{k+1}}(b_2), \cdots, \eta_{\mathcal{A}_{k+1}}(b_m)\} \\ &\leq \min\{\eta_{\mathcal{A}_{k+1}}(b_1), \eta_{\mathcal{A}_{k+1}}(b_2), \cdots, \eta_{\mathcal{A}_{k+1}}(b_m)\} \\ &\leq \min\{\eta_{\mathcal{A}_{k+1}}(a_1 b_1), \eta_{\mathcal{A}_{k+1}}(a_2 b_2), \cdots, \eta_{\mathcal{A}_{k+1}}(a_m b_m)\} \\ &\leq \eta_{\mathcal{A}_{k+1}}(a_1 b_1 + a_2 b_2 + \cdots + a_m b_m). \end{aligned}$$

Thus we get

(5.9)

$$\min\{\eta_{\mathcal{A}_1 * \mathcal{A}_2 * \cdots * \mathcal{A}_k}(a_1), \cdots, \eta_{\mathcal{A}_1 * \mathcal{A}_2 * \cdots * \mathcal{A}_k}(a_m), \eta_{\mathcal{A}_{k+1}}(b_1), \eta_{\mathcal{A}_{k+1}}(b_2), \cdots, \eta_{\mathcal{A}_{k+1}}(b_m) \leq \eta_{\mathcal{A}_{k+1}}(n).$$

Since $\eta_{\mathcal{A}_i}$ is fuzzy ideal for all $1 \leq i \leq p$, we have

$$\begin{aligned} &\min\{\eta_{\mathcal{A}_1 * \mathcal{A}_2 * \cdots * \mathcal{A}_k}(a_1), \eta_{\mathcal{A}_1 * \mathcal{A}_2 * \cdots * \mathcal{A}_k}(a_2), \cdots, \eta_{\mathcal{A}_1 * \mathcal{A}_2 * \cdots * \mathcal{A}_k}(a_m), \\ &\qquad \eta_{\mathcal{A}_{k+1}}(b_1), \eta_{\mathcal{A}_{k+1}}(b_2), \cdots, \eta_{\mathcal{A}_{k+1}}(b_m)\} \\ &\leq \min\{\eta_{\mathcal{A}_1 * \mathcal{A}_2 * \cdots * \mathcal{A}_k}(a_1), \eta_{\mathcal{A}_1 * \mathcal{A}_2 * \cdots * \mathcal{A}_k}(a_2), \cdots, \eta_{\mathcal{A}_1 * \mathcal{A}_2 * \cdots * \mathcal{A}_k}(a_m)\} \\ &\leq \min[\min\{\eta_{\mathcal{A}_1}(a_1), \eta_{\mathcal{A}_2}(a_1), \cdots, \eta_{\mathcal{A}_k}(a_1)\}, \cdots, \min\{\eta_{\mathcal{A}_1}(a_m), \eta_{\mathcal{A}_2}(a_m), \cdots, \eta_{\mathcal{A}_k}(a_m)\}] \\ &\leq \min[\min\{\eta_{\mathcal{A}_1}((0 + a_1)b_1 - 0b_1), \cdots, \eta_{\mathcal{A}_k}((0 + a_1)b_1 - 0b_1)\}, \cdots, \min\{\eta_{\mathcal{A}_1}((0 + a_m)b_m - 0b_m), \cdots, \eta_{\mathcal{A}_k}((0 + a_m)b_m - 0b_m)\}] \\ &\leq \min[\min\{\eta_{\mathcal{A}_1}(a_1 b_1), \cdots, \eta_{\mathcal{A}_k}(a_1 b_1)\}, \cdots, \min\{\eta_{\mathcal{A}_1}(a_m b_m), \cdots, \eta_{\mathcal{A}_k}(a_m b_m)\}] \\ &\leq \min\{\eta_{\mathcal{A}_1}(a_1 b_1 + a_2 b_2 + \cdots + a_m b_m), \eta_{\mathcal{A}_2}(a_1 b_1 + a_2 b_2 + \cdots + a_m b_m), \cdots, \eta_{\mathcal{A}_k}(a_1 b_1 + a_2 b_2 + \cdots + a_m b_m)\}, \end{aligned}$$

i.e.,

(5.10)

$$\min\{\eta_{\mathcal{A}_1 * \cdots * \mathcal{A}_k}(a_1), \cdots, \eta_{\mathcal{A}_1 * \cdots * \mathcal{A}_k}(a_m), \eta_{\mathcal{A}_{k+1}}(b_1), \eta_{\mathcal{A}_{k+1}}(b_2), \cdots, \eta_{\mathcal{A}_{k+1}}(b_m)\} \leq \min\{\eta_{\mathcal{A}_1}(n), \cdots, \eta_{\mathcal{A}_k}(n)\}.$$

$$\begin{aligned} \eta_{\mathcal{A}_1 * \mathcal{A}_2 * \cdots * \mathcal{A}_k * \mathcal{A}_{k+1}}(n) &\leq \min\{\eta_{\mathcal{A}_1 * \mathcal{A}_2 * \cdots * \mathcal{A}_k}(n), \eta_{\mathcal{A}_{k+1}}(n)\} \\ &\leq \min[\min\{\eta_{\mathcal{A}_1}(n), \eta_{\mathcal{A}_2}(n), \cdots, \eta_{\mathcal{A}_k}(n)\}, \eta_{\mathcal{A}_{k+1}}(n)] \\ &\leq \min\{\eta_{\mathcal{A}_1}(n), \eta_{\mathcal{A}_2}(n), \cdots, \eta_{\mathcal{A}_k}(n), \eta_{\mathcal{A}_{k+1}}(n)\}. \end{aligned}$$

It follows from the equation (5.10),

$$\begin{aligned} \eta_{(\mathcal{A}_1 * \mathcal{A}_2 * \cdots * \mathcal{A}_k) * \mathcal{A}_{k+1}}(n) &= \bigvee_{n = \sum\limits_{finite} a_i b_i} \min \begin{cases} \eta_{\mathcal{A}_1 * \mathcal{A}_2 * \cdots * \mathcal{A}_k}(a_1), \eta_{\mathcal{A}_1 * \mathcal{A}_2 * \cdots * \mathcal{A}_k}(a_2), \cdots, \eta_{\mathcal{A}_1 * \mathcal{A}_2 * \cdots * \mathcal{A}_k}(a_m) \\ \eta_{\mathcal{A}_{k+1}}(b_1), \eta_{\mathcal{A}_{k+1}}(b_2), \cdots, \eta_{\mathcal{A}_{k+1}}(b_m) \end{cases} \\ &\leq \min \begin{cases} \eta_{\mathcal{A}_1 * \mathcal{A}_2 * \cdots * \mathcal{A}_k}(a_1), \eta_{\mathcal{A}_1 * \mathcal{A}_2 * \cdots * \mathcal{A}_k}(a_2), \cdots, \eta_{\mathcal{A}_1 * \mathcal{A}_2 * \cdots * \mathcal{A}_k}(a_m) \\ \eta_{\mathcal{A}_{k+1}}(b_1), \eta_{\mathcal{A}_{k+1}}(b_2), \cdots, \eta_{\mathcal{A}_{k+1}}(b_m) \end{cases} \\ &\leq \min\{\eta_{\mathcal{A}_1}(n), \eta_{\mathcal{A}_2}(n), \cdots, \eta_{\mathcal{A}_k}(n)\} \end{aligned}$$

i.e.,

$$\eta_{(\mathcal{A}_1*\mathcal{A}_2*\cdots*\mathcal{A}_k)*\mathcal{A}_{k+1}}(n) \leq \min\{\eta_{\mathcal{A}_1}(n), \eta_{\mathcal{A}_2}(n), \cdots, \eta_{\mathcal{A}_k}(n)\}. \tag{5.11}$$

In the similar manner

$$\eta_{(\mathcal{A}_1*\mathcal{A}_2*\cdots*\mathcal{A}_k)*\mathcal{A}_{k+1}}(n) \leq \eta_{\mathcal{A}_{k+1}}(n). \tag{5.12}$$

By equations (5.11) and (5.12),

$$\eta_{(\mathcal{A}_1*\mathcal{A}_2*\cdots*\mathcal{A}_k)*\mathcal{A}_{k+1}}(n) \leq \min[\min\{\eta_{\mathcal{A}_1}(n), \eta_{\mathcal{A}_2}(n), \cdots, \eta_{\mathcal{A}_k}(n)\}, \eta_{\mathcal{A}_{k+1}}(n)]. \tag{5.13}$$

Also,

$$\begin{aligned}
&\max\{\delta_{\mathcal{A}_1*\mathcal{A}_2*\cdots*\mathcal{A}_k}(a_1), \delta_{\mathcal{A}_1*\mathcal{A}_2*\cdots*\mathcal{A}_k}(a_2), \cdots, \delta_{\mathcal{A}_1*\mathcal{A}_2*\cdots*\mathcal{A}_k}(a_m), \delta_{\mathcal{A}_{k+1}}(b_1), \eta_{\mathcal{A}_{k+1}}(b_2), \cdots, \delta_{\mathcal{A}_{k+1}}(b_m)\}\\
&\geq \max\{\delta_{\mathcal{A}_1*\mathcal{A}_2*\cdots*\mathcal{A}_k}(a_1), \delta_{\mathcal{A}_1*\mathcal{A}_2*\cdots*\mathcal{A}_k}(a_2), \cdots, \delta_{\mathcal{A}_1*\mathcal{A}_2*\cdots*\mathcal{A}_k}(a_m)\}\\
&\geq \max[\max\{\delta_{\mathcal{A}_1}(a_1), \delta_{\mathcal{A}_2}(a_1), \cdots, \delta_{\mathcal{A}_k}(a_1)\}, \cdots, \max\{\delta_{\mathcal{A}_1}(a_m), \delta_{\mathcal{A}_2}(a_m), \cdots, \delta_{\mathcal{A}_k}(a_m)\}]\\
&\geq \max[\max\{\delta_{\mathcal{A}_1}((0+a_1)b_1 - 0b_1), \cdots, \delta_{\mathcal{A}_k}((0+a_1)b_1 - 0b_1)\}, .., \max\{\delta_{\mathcal{A}_1}((0+a_m)b_m - 0b_m), ..\\
&\quad .., \delta_{\mathcal{A}_k}((0+a_m)b_m - 0b_m)\}]\\
&\geq \max[\max\{\delta_{\mathcal{A}_1}(a_1b_1), ..., \delta_{\mathcal{A}_k}(a_1b_1)\}, .., \delta_{\mathcal{A}_1}(a_mb_m), .., \delta_{\mathcal{A}_k}(a_mb_m)]\\
&\geq \max[\max\{\delta_{\mathcal{A}_1}(a_1b_1), \delta_{\mathcal{A}_1}(a_2b_2), ..., \delta_{\mathcal{A}_k}(a_1b_1)\}, .., \delta_{\mathcal{A}_1}(a_mb_m), .., \delta_{\mathcal{A}_k}(a_mb_m)]\\
&\geq \max\{\delta_{\mathcal{A}_1}(a_1b_1 + a_2b_2 + \cdots + a_mb_m), \delta_{\mathcal{A}_2}(a_1b_1 + a_2b_2 + \cdots + a_mb_m), .., \delta_{\mathcal{A}_k}(a_1b_1 + a_2b_2\\
&\quad + \cdots + a_mb_m)\}.
\end{aligned}$$

i.e.,

(5.14)

$$\begin{aligned}
&\max\{\delta_{\mathcal{A}_1*\mathcal{A}_2*\cdots*\mathcal{A}_k}(a_1), \delta_{\mathcal{A}_1*\mathcal{A}_2*\cdots*\mathcal{A}_k}(a_2), ..., \delta_{\mathcal{A}_1*\mathcal{A}_2*\cdots*\mathcal{A}_k}(a_m), \delta_{\mathcal{A}_{k+1}}(b_1), \delta_{\mathcal{A}_{k+1}}(b_2), ..., \delta_{\mathcal{A}_{k+1}}(b_m)\}\\
&\geq \max\{\delta_{\mathcal{A}_1}(n), \delta_{\mathcal{A}_2}(n), .., \delta_{\mathcal{A}_k}(n)\}.
\end{aligned}$$

Now,

$$\begin{aligned}
\delta_{(\mathcal{A}_1*\mathcal{A}_2*\cdots*\mathcal{A}_k)*\mathcal{A}_{k+1}}(n) &= \bigwedge_{n=\sum_{finite} a_ib_i} \max\begin{cases}\delta_{\mathcal{A}_1*\mathcal{A}_2*\cdots*\mathcal{A}_k}(a_1), \delta_{\mathcal{A}_1*\mathcal{A}_2*\cdots*\mathcal{A}_k}(a_2), ..., \delta_{\mathcal{A}_1*\mathcal{A}_2*\cdots*\mathcal{A}_k}(a_m),\\ \delta_{\mathcal{A}_{k+1}}(b_1), \delta_{\mathcal{A}_{k+1}}(b_2), ..., \delta_{\mathcal{A}_{k+1}}(b_m),\end{cases}\\
&\geq \max\begin{cases}\delta_{\mathcal{A}_1*\mathcal{A}_2*\cdots*\mathcal{A}_k}(a_1), \delta_{\mathcal{A}_1*\mathcal{A}_2*\cdots*\mathcal{A}_k}(a_2), ..., \delta_{\mathcal{A}_1*\mathcal{A}_2*\cdots*\mathcal{A}_k}(a_m),\\ \delta_{\mathcal{A}_{k+1}}(b_1), \delta_{\mathcal{A}_{k+1}}(b_2), ..., \delta_{\mathcal{A}_{k+1}}(b_m),\end{cases}\\
&\geq \max\{\delta_{\mathcal{A}_1}(n), \delta_{\mathcal{A}_2}(n), ..., \delta_{\mathcal{A}_k}(n)\}
\end{aligned}$$

i.e.,

$$\delta_{(\mathcal{A}_1*\mathcal{A}_2*\cdots*\mathcal{A}_k)*\mathcal{A}_{k+1}}(n) \geq \max\{\delta_{\mathcal{A}_1}(n), \delta_{\mathcal{A}_2}(n), ..., \delta_{\mathcal{A}_k}(n)\} \tag{5.15}$$

and

$$\delta_{(\mathcal{A}_1*\mathcal{A}_2*\cdots*\mathcal{A}_k)*\mathcal{A}_{k+1}}(n) \geq \delta_{\mathcal{A}_{k+1}}(n). \tag{5.16}$$

So, by equations (5.15) and (5.16)

$$\begin{aligned}
\delta_{(\mathcal{A}_1*\mathcal{A}_2*\cdots*\mathcal{A}_k)*\mathcal{A}_{k+1}}(n) &\geq \max[\max\{\delta_{\mathcal{A}_1}(n), \delta_{\mathcal{A}_2}(n), ..., \delta_{\mathcal{A}_k}(n)\}, \delta_{\mathcal{A}_{k+1}}(n)]\\
&\geq \max\{\delta_{\mathcal{A}_1}(n), \delta_{\mathcal{A}_2}(n), ..., \delta_{\mathcal{A}_k}(n), \delta_{\mathcal{A}_{k+1}}(n)\}
\end{aligned}$$

$$\delta_{(\mathcal{A}_1*\mathcal{A}_2*\cdots*\mathcal{A}_k)*\mathcal{A}_{k+1}}(n) \geq \max\{\delta_{\mathcal{A}_1}(n), \delta_{\mathcal{A}_2}(n), ..., \delta_{\mathcal{A}_k}(n), \delta_{\mathcal{A}_{k+1}}(n)\}. \tag{5.17}$$

From equations (5.13) and (5.17)

$$\mathcal{A}_1 * \mathcal{A}_2 * \cdots * \mathcal{A}_p \subseteq \mathcal{A}_1 \cap \mathcal{A}_2 \cap \cdots \cap \mathcal{A}_p.$$

□

Theorem 5.4. *If $\mathcal{Q}$ is an IF prime ideal of $\mathcal{N}$. Then $\mathcal{Q}^{\mathcal{G}}$ is an IF $\mathcal{G}$-prime ideal of $\mathcal{N}$. Conversely, if $\mathcal{P}$ is an IF $\mathcal{G}$-prime ideal of $\mathcal{N}$, then there exists an IF prime ideal $\mathcal{Q}$ of $\mathcal{N}$ such that $\mathcal{Q}^{\mathcal{G}} = \mathcal{P}$, where $\mathcal{Q}$ is unique up to its $\mathcal{G}$-orbit.*

Proof. Suppose that $\mathcal{Q}$ is an IF prime ideal of $\mathcal{N}$ and $\mathcal{Q}_1, \mathcal{Q}_2$ are $\mathcal{G}$-invariant IF ideals of $\mathcal{N}$ such that $\mathcal{Q}_1 * \mathcal{Q}_2 \subseteq \mathcal{Q}^{\mathcal{G}}$. Since $\mathcal{Q}^{\mathcal{G}} \subseteq \mathcal{Q}$, it follows that $\mathcal{Q}_1 * \mathcal{Q}_2 \subseteq \mathcal{Q}$. Then the primeness of $\mathcal{Q}$ implies that either $\mathcal{Q}_1 \subseteq \mathcal{Q}$ or $\mathcal{Q}_2 \subseteq \mathcal{Q}$. By Theorem 3.7, either $\mathcal{Q}_1 \subseteq \mathcal{Q}^{\mathcal{G}}$ or $\mathcal{Q}_2 \subseteq \mathcal{Q}^{\mathcal{G}}$. Thus $\mathcal{Q}^{\mathcal{G}}$ is IF $\mathcal{G}$-prime ideal.

Conversely, suppose that $\mathcal{P}$ is an IF $\mathcal{G}$-prime ideal of $\mathcal{N}$ and consider

$$\mathcal{T} = \{\mathcal{A}, \text{an IF ideal of } \mathcal{N} | \mathcal{A}^{\mathcal{G}} \subseteq \mathcal{P}\}.$$

Since $\mathcal{P}^{\mathcal{G}} \subseteq \mathcal{P}$, then $\mathcal{T}$ is not empty. Let $\mathcal{C} = \{\mathcal{A}_i\}$ be chain of IF ideals of $\mathcal{N}$ in T. Then $\cup\mathcal{A}_i$ is an IF ideal of $\mathcal{N}$. Furthermore, $(\cup\mathcal{A}_i)^{\mathcal{G}} \subseteq \mathcal{P}$. Let $\cup\mathcal{A}_i = (\eta, \delta)$ and $\mathcal{P} = (\eta_{\mathcal{P}}, \delta_{\mathcal{P}})$. Then $\eta = \bigvee_i \eta_{\mathcal{A}_i}$ and $\delta = \bigwedge_i \delta_{\mathcal{A}_i}$, where $\mathcal{A}_i = (\eta_{\mathcal{A}_i}, \delta_{\mathcal{A}_i})$. For every $n \in \mathcal{N}$,

$$\begin{aligned}\eta^{\mathcal{G}}(n) &= \min_{g\in\mathcal{G}}(\bigvee_i(\eta_{\mathcal{A}_i}(n^g)))\\ &= \bigvee_i(\min_{g\in\mathcal{G}}(\eta_{\mathcal{A}_i}(n^g)))\\ &\leq \eta_{\mathcal{P}}(n),\end{aligned}$$

since each $\mathcal{A}_i^{\mathcal{G}} \subseteq \mathcal{P}$. Also

$$\begin{aligned}\delta^{\mathcal{G}}(n) &= \max_{g\in\mathcal{G}}(\bigwedge_i(\delta_{\mathcal{A}_i}(n^g)))\\ &= \bigwedge_i(\max_{g\in\mathcal{G}}(\delta_{\mathcal{A}_i}(n^g)))\\ &\geq \delta_{\mathcal{P}}(n),\end{aligned}$$

By Zorn's lemma, let $\mathcal{Q}$ be a maximal IF ideal and $\mathcal{B}_1$, $\mathcal{B}_2$ are IF ideals of $\mathcal{N}$ such that $\mathcal{B}_1 * \mathcal{B}_2 \subseteq \mathcal{Q}$. Then $(\mathcal{B}_1 * \mathcal{B}_2)^{\mathcal{G}} \subseteq \mathcal{Q}^{\mathcal{G}} \subset \mathcal{P}$, since $\mathcal{B}_1^{\mathcal{G}} * \mathcal{B}_2^{\mathcal{G}} \subseteq \mathcal{B}_1 * \mathcal{B}_2$ and $\mathcal{B}_1^{\mathcal{G}} * \mathcal{B}_2^{\mathcal{G}}$ is $\mathcal{G}-$ invariant,

$$\mathcal{B}_1^{\mathcal{G}} * \mathcal{B}_2^{\mathcal{G}} \subseteq (\mathcal{B}_1 * \mathcal{B}_2)^{\mathcal{G}}.$$

Thus $\mathcal{B}_1^{\mathcal{G}} * \mathcal{B}_2^{\mathcal{G}} \subseteq (\mathcal{B}_1 * \mathcal{B}_2)^{\mathcal{G}} \subseteq \mathcal{P}$ implies $\mathcal{B}_1^{\mathcal{G}} \subseteq \mathcal{P}$ or $\mathcal{B}_2^{\mathcal{G}} \subseteq \mathcal{P}$. By the maximality of $\mathcal{Q}$, either $\mathcal{B}_1 \subseteq \mathcal{Q}$ or $\mathcal{B}_2 \subseteq \mathcal{Q}$. So $\mathcal{Q}$ is IF prime ideal. Since $\mathcal{P} \subseteq \mathcal{Q}, \mathcal{P}^{\mathcal{G}} \subseteq \mathcal{Q}^{\mathcal{G}}$, $\mathcal{Q}^{\mathcal{G}} = \mathcal{P}$.

Let there exist another IF prime ideal $\mathcal{M}$ of $\mathcal{N}$ such that $\mathcal{M}^{G} = \mathcal{P}$. Also $\underset{g\in\mathcal{G}}{*}\mathcal{Q}^g \subseteq \underset{g\in\mathcal{G}}{\cap}\mathcal{Q}^g = \mathcal{Q}^{\mathcal{G}} \subseteq \mathcal{M}$, $\mathcal{Q}^g \subseteq \mathcal{M}$ for some $g \in \mathcal{G}$ by the primeness of $\mathcal{M}$. Then $\mathcal{Q} \subseteq \mathcal{M}^{g^{-1}}$ and $\mathcal{Q} = \mathcal{M}^{g^{-1}}$ since $(\mathcal{M}^{g^{-1}})^{\mathcal{G}} = \mathcal{M}^{\mathcal{G}} \subseteq \mathcal{P}$ implies that $\mathcal{M}$ is contained in the set T. Thus $\mathcal{Q}$ is unique upto $\mathcal{G}$- orbits. □

6. Conclusion

The exploration of IF prime ideals of near rings is an evolving field of study with many potential applications. In this manuscript, we have introduced and investigated group action on IF ideals of near ring. Furthermore, we have shown that how an IF prime ideal is related to IF $\mathcal{G}$-prime ideal of $\mathcal{N}$. In future, we may extend this work for more general structure like picture fuzzy ideal of near ring, intuitionistic fuzzy ideal of semi rings.

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References

[1] L .A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.
[2] K. T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1986) 87–96.
[3] Wang-jin Liu, Fuzzy Invariant Subgroups and Fuzzy Ideals, Fuzzy Sets and Systems 8 (1982) 133–139.
[4] Seung Dong Kim and Hee Sik Kim, On Fuzzy Ideals of Near-Rings, Bull. Korean Math. Soc. 33 (1996) 593–601.
[5] Ranjit Biswas, Intuitionistic Fuzzy Subgroup, Notes on Intuitionistic Fuzzy Sets 3 (1997) 53–60.
[6] K. Hur, S. Y. Jang and H. W. Kang, Intuitionistic fuzzy ideals of a ring, J. Korea Soc.Math.Educ.Ser. B: Pure Appl.Math. 12 (2005) 193–209.
[7] Y. B. Jun and C. H. Park, Intrinsic product of intuitionistic fuzzy subrings/ideals in a ring, Honam Math.J. 28 (2006) 439–469.
[8] Zhan Jianming and MA Xueling, Intuitionistic fuzzy ideals of near ring, Scientiae Mathematicae Japonicae (2004) 289–293.
[9] Asma Ali, R. P. Sharma and Arshad Zishan, Group Action on Fuzzy Ideals of Near Rings, In: Sharma, R.K., Pareschi, L., Atangana, A., Sahoo, B., Kukreja, V.K. (eds) Frontiers in Industrial and Applied Mathematics. FIAM 2021. Springer Proceedings in Mathematics & Statistics, vol 410. Springer (2023) 347–367.
[10]] Dong Soo Lee and Chul Hwan Park, Group action on intuitionistic fuzzy ideals of rings, East Asian Math. 22 (2006) 239–248.
[11] K. H. Kim and Y. H. Jun On Anti Fuzzy ideals of near rings, Iran. J. Fuzzy Syst. 2 (2005) 71–80.
[12] P. B. Bhattacharya, S. K. Jain and S. R. Nagpaul Basic Abstract Algebra, Cambridge University Press (2006).

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